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On the miracle of the Coleman-Glashow and other baryon mass formulas

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Abstract. Due to a new measurement of the Ξ^0 mass, the Coleman-Glashow formula for the baryon octet e.m. masses (derived using unbroken flavor SU_3) is satisfied to an extraordinary level of precision. The same unexpected precision exists for the Gell Mann-Okubo formula and for its octet-decuplet extension (G. Morpurgo, Phys. Rev. Lett. 68 (1992) 139). We show that the old question "why do they work so well?" is now answered by the general parametrization method. (*PACS:* 12.38.Aw; 13.40.Dk; 14.20.-c)

1. Introduction

A recent measurement of the Ξ^0 mass [1] lowered considerably its error. The Ξ^0 mass is now $1314.82 \pm 0.06 \pm 0.2$ MeV. The importance of a new measurement was noted long ago [2] in connection with the Coleman-Glashow (CG) e.m. mass formula. The previous value was 1314.9 ± 0.6 MeV [3]. Indeed now the agreement of CG with the data is more miracolous than ever. Writing the CG formula as:

$$p - n = \Sigma^+ - \Sigma^- + \Xi^- - \Xi^0 \tag{1}$$

the present data give:

$$l.h.s. = -1.29 \text{ MeV} \qquad\qquad r.h.s. = -1.58 \pm 0.25 \text{ MeV} \tag{2}$$

Because the mass differences $\Sigma^- - \Sigma^+$ in (1) is ≈ 8 MeV, the agreement is amazing ($\cong (4 \pm 3)\%$) [before [1], it was already excellent [2] (1.29 to be compared with 1.67 ± 0.6)]. To appreciate the point, note that the CG formula was derived [4] assuming unbroken flavor SU(3); but flavor is violated -in the baryon octet- by $\approx 33\%$.

A similar situation applies to the Gell Mann-Okubo mass formula and its octet-decuplet extension by one of us [5]. It also holds -with larger errors- for some formulas of Gal and Scheck [6]. We already discussed [2] all these relations using the QCD general parametrization method, but the result [1] suggests a revisitation. Indeed we are dealing perhaps with one of the most precise estimates in processes where the strong interactions play a role.

As stated above, the original derivation of CG neglected entirely the flavor breaking of the strong interactions. But it was shown in [2] that the CG formula can be derived also keeping *all the flavor breaking terms*, with the only omission of terms with 3-quark indices. Here we complete the derivation [2]; we include, in addition to the terms considered in [2], the effect of the $m_d - m_u$ mass difference and the so called Trace terms, absent in [2]; they do not alter the conclusions of [2].

2. A brief summary of the general parametrization method

It is convenient to recall briefly the QCD parametrization method [7, 8]. The method, based only on general features of QCD, applies to a variety of QCD matrix elements or expectation values. By integrating on all internal $q\bar{q}$ and gluon lines, the method parametrizes exactly such matrix elements. Thus hadron properties -like e.m. masses, including their flavor-breaking contributions- are written exactly as a sum of some spin-flavor structures each multiplied by a coefficient. Each structure (term) has, for baryons, a maximum of three indices. The coef-

ficients of the various terms are seen to decrease with increasing complexity of the term. By the way this "hierarchy" explains why the non relativistic quark model (NRQM), that keeps only the simplest terms, works quantitatively fairly well. Though the parametrization is performed in a given Lorentz frame and is, therefore non covariant, it is relativistic, being derived exactly from a relativistic field theory, QCD. For the basis of the method see [7, 8]; applications are also given in [2,5,7-11]. Other references are listed in [11, 12]; the latter gives a short review.

Here we will not recall the details of the method, but -for completeness- summarize it. The e.m. contribution to the mass of a baryon B is:

$$\langle\psi_B|\Omega|\psi_B\rangle = \langle\phi_B|V^\dagger\Omega V|\phi_B\rangle \tag{3}$$

In (3) Ω is -to second order in the charge- the exact QCD operator, expressed in terms of quark fields, representing the e.m. contribution to the mass, including all flavor breaking contributions of the strong interactions; $|\psi_B\rangle$ is the exact eigenstate of B at rest of the QCD Hamiltonian; $|\phi_B\rangle$ is an auxiliary three body state of B, factorizable as

$$|\phi_B\rangle = |X_{L=0}\cdot W_B\rangle \tag{4}$$

into a space part $X_{L=0}$ with orbital angular momentum zero and a spin unitary-spin part W_B. The unitary transformation V -applied to the auxiliary state $|\phi_B\rangle$- transforms the latter into $|\psi_B\rangle$. After integration on the space variables, (4) can be written

$$\langle\psi_B|\Omega|\psi_B\rangle = \langle\phi_B|V^\dagger\Omega V|\phi_B\rangle = \langle W_B|\sum_\nu t_\nu\Gamma_\nu(s,f)|W_B\rangle \tag{5}$$

where Γ_ν are operators depending only on the spin and flavor variables of the three quarks in ϕ_B and the t_ν's are a set of parameters. Of course [7] the operator V dresses the auxiliary state $|\phi_B\rangle$ with $q\bar{q}$ pairs and gluons and also introduces

configuration mixing. Thus it generates the exact QCD eigenstate $|\psi_B\rangle$. It is the factorizability of $|\phi_B\rangle$ that allows the second step in Eq.(5), eliminating the space coordinates. The list of $\Gamma_\nu(s,f)$'s in (5) for the e.m. masses was given in [2]. As stated, we will revisit this parametrization.

3. The corrections to the octet masses in the Coleman-Glashow formula

The e.m. corrections to the octet baryon masses including flavor breaking analyzed in [2] left out the $m_d - m_u$ contribution and the Trace terms that, in this paper, we will include later (Sect. 4). For the moment we reanalyze the results of [2]. There we called $\delta_i B$ the e.m. mass correction of baryon B ($i = 0, 1, 2$ refers to the order of the s-flavor breaking). Thus by $\delta_0 B$ we meant (and mean) the e.m. correction neglecting all flavor breaking effects, $\delta_1 B$ is the e.m. correction including only first order flavor breaking effects and so on for $\delta_2 B$.

The quantities $\Gamma_\nu(s,f)$ that enter in the construction of $\delta_i B$'s are given in Eqs. (17-19) of [2]. There is nothing to change in these equations except for a (trivial) point of notation. In [2] the strange quark was called λ (the non strange ones $\mathcal{N}$ and $\mathcal{P}$). Here we use the standard current notation s, d, u. We do this because (see [8]) we can select as we like the q^2 of the renormalization point of the quark mass and we now turn to the standard q around 1 GeV. Thus the projectors P^λ, $P^\mathcal{N}$, $P^\mathcal{P}$ in [2] are now written P^s, P^d, P^u.

The Γ_ν's in Eqs.(6,7) below are the same as those in Eqs.(17, 18) of [2] (the sum symbol in each Γ_ν is defined in [2]); we transcribe them here:

1) Γ's of *zero order* in flavor breaking:

$$\begin{array}{lll} \Gamma_1 = \sum[Q_i^2] & ; \quad \Gamma_2 = \sum[Q_i^2(\boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_k)] & ; \quad \Gamma_3 = \sum[Q_i^2(\boldsymbol{\sigma}_j \cdot \boldsymbol{\sigma}_k)] \\ \Gamma_4 = \sum[Q_i Q_k] & ; \quad \Gamma_5 = \sum[Q_i Q_k(\boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_k)] & ; \quad \Gamma_6 = \sum[Q_i Q_k(\boldsymbol{\sigma}_i + \boldsymbol{\sigma}_k)\cdot \sigma_j] \end{array} \tag{6}$$

2) Γ's of *first order* in P^s (acting in Λ,Σ,Σ^*,Ξ,Ξ^*,Ω):

$$\begin{array}{lclcl}
\Gamma_7 = \sum[Q_i^2 P_i^s] & ; & \Gamma_8 = \sum[Q_i^2 P_i^s(\boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_k)] & ; & \Gamma_9 = \sum[Q_i^2 P_i^s(\boldsymbol{\sigma}_j \cdot \boldsymbol{\sigma}_k)] \\
\Gamma_{10} = \sum[Q_i^2 P_k^s] & ; & \Gamma_{11} = \sum[Q_i^2 P_k^s(\boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_k)] & ; & \Gamma_{12} = \sum[Q_i^2 P_k^s((\boldsymbol{\sigma}_i + \boldsymbol{\sigma}_k) \cdot \boldsymbol{\sigma}_j)] \\
\Gamma_{13} = \sum[Q_i Q_k P_i^s] & ; & \Gamma_{14} = \sum[Q_i Q_k P_i^s(\boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_k)] & ; & \Gamma_{15} = \sum[Q_i Q_k P_i^s(\boldsymbol{\sigma}_i + \boldsymbol{\sigma}_k) \cdot \boldsymbol{\sigma}_j] \\
\Gamma_{16} = \sum[Q_i Q_k P_j^s] & ; & \Gamma_{17} = \sum[Q_i Q_k P_j^s(\boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_k)] & ; & \Gamma_{18} = \sum[Q_i Q_k P_j^s(\boldsymbol{\sigma}_i + \boldsymbol{\sigma}_k) \cdot \boldsymbol{\sigma}_j]
\end{array} \tag{7}$$

where $Q_i = \frac{2}{3}P_i^u - \frac{1}{3}P_i^d - \frac{1}{3}P_i^s$ and the P_i's are the projectors on the u, d, s quarks.

As to the Γ_ν's of second and third order in P^s, they are listed in Eqs. (19,20) of [2]. We will not transcribe them here.

The expression $\delta_0 B$ of the electromagnetic mass of B at zero order in flavor breaking is:

$$\delta_0 B = a\Gamma_1 + b\Gamma_2 + c\Gamma_3 + d\Gamma_4 + e\Gamma_5 + f\Gamma_6 \tag{8}$$

(where, to agree with [2], we used $a \ldots f$ instead of $t_1 \ldots t_6$). As shown in [2] one can check that:

$$\delta_0 P - \delta_0 N = \delta_0 \Sigma^+ - \delta_0 \Sigma^- + \delta_0 \Xi^- - \delta_0 \Xi^0$$

which is the CG relation at zero order in flavor breaking.

As to $\delta_1 B$ and $\delta_2 B$, coming from the first and second order in flavor breaking, we refer to [2]. There it is shown (table III) that, except for terms with three indices, also $\delta_1 B$ and $\delta_2 B$ leave the CG relation unaltered. Thus, to *all orders in flavor breaking, with the only omission of three quark terms in $\delta_1 B$ and $\delta_2 B$*, the CG relation holds:

$$\delta P - \delta N = \delta \Sigma^+ - \delta \Sigma^- + \delta \Xi^- - \delta \Xi^0 \tag{9}$$

where now $\delta \equiv \delta_0 + \delta_1 + \delta_2$.

To evaluate the order of magnitude of the three quark terms of the type $\delta_1 B$ and $\delta_2 B$ (the only ones that violate the CG formula) we now use the hierarchy discussed at length in previous papers [9, 8]. Of course, the dominant order of magnitude, is here that of the two index terms of the form $Q_i Q_k$. Let us consider the three quark terms in $\delta_1 B$. One can show that Γ_9, Γ_{12} and Γ_{15} do not contribute

to the left and right hand sides of the CG formula. As to Γ_{16}, Γ_{17} and Γ_{18}, they do not contribute to $\delta_1 n$ and $\delta_1 p$, while for $\delta_1\Sigma^+ - \delta_1\Sigma^- + \delta_1\Xi^- - \delta_1\Xi^0$ one gets in all cases ("*something*")/3. As to the magnitude of "*something*", note that according to the hierarchy, each of Γ_{16}, Γ_{17} and Γ_{18} carries a reduction factor ≈ 0.3 due to the presence of a P^s and ≈ 0.3 due to one more gluon exchange. Thus we have a reduction of order $(1/3)^3 \cong 4 \cdot 10^{-2}$ for each of the above terms with respect to the dominant no flavor-breaking contributions. (Note: Γ_{18} here is 1/2 that in [2] due to an incorrect normalization in [2] that produced, however, no effect because 3-index terms were not evaluated in [2].)

Because experimentally $\Sigma^- - \Sigma^+ \cong 8$ MeV and $\Xi^- - \Xi^0 \cong 6.4$ MeV, the above value of $4 \cdot 10^{-2}$ implies an expected difference between left and right hand sides of the CG formula of $\approx 0.2 \div 0.3$ MeV that does not disagree with the data.

A similar argument holds for the three-quark terms of second order in flavor breaking listed as (19) in [2]. These second order flavor breaking three-quark terms, are expected from the hierarchy, to be $\approx$ (1/9) of the first order flavor breaking terms mentioned above; that is

$$|\delta_2\Xi^- - \delta_2\Xi^0| \approx \frac{1}{9}|\delta_1\Sigma^+ - \delta_1\Sigma^- + \delta_1\Xi^- - \delta_1\Xi^0|$$

and therefore the three quark, second order flavor breaking effect should contribute to the difference between left and right hand side of the CG relation by ≈ 0.02 MeV.

4. The effects of $m_d - m_u$ and of the Trace terms on the CG relation

a) *The effect of* $m_d - m_u$. As shown in [8] the quantity that intervenes in evaluating the effect of a mass difference Δm between quarks is $\Delta m/(\beta\Lambda)$ where $\Lambda \equiv \Lambda_{QCD} \cong 200$ MeV and β is approximately 3. Because Δm for d and u is a few MeV, only the first order term in the expansion in $\Delta m/(\beta\Lambda)$ may be relevant.

But it is easy to check that the CG formula is not modified. (Obviously product terms of Δm and $Q_i Q_k$ type perturbations are totally negligible.)

b) *The effect of the Trace terms.* In addition to the terms in [2] other terms are present in the general parametrization (see [8], in particular footnote 14). They leave the CG formula unaltered, as we will show; they must however be recorded. These "Trace" terms correspond to QCD Feynman closed loops, as exemplified, e.g. in [10] (fig.1) or [12] (fig.3). Introducing the matrix

$$Q = \frac{2}{3}P^u - \frac{1}{3}P^d - \frac{1}{3}P^s \tag{10}$$

(not to be confused with the baryon charge Q_B that was called Q in [2]), the Trace terms can be constructed as follows: Consider the quantities

$$T_1 = Tr(QP^s) \ , \qquad T_2 = Tr(Q^2) \ , \qquad T_3 = Tr(Q^2 P^s)$$

and multiply them by $\sum(\boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_k)$ or by the sums listed in Eqs.(6) and (7) keeping only those expressions that, after the multiplication, contain two Q symbols (either Q^2 or $Q \cdot Q_i$ or $Q_i Q_k$) and a number of P^s from 0 to 2; for instance (just to exemplify)

$$\begin{array}{lll} Tr(Q^2)\sum(\boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_k) \ , & Tr(Q^2 P^s)\sum(\boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_k) \ , & Tr(QP^s)\sum Q_i \ , \\ Tr(QP^s)\sum Q_i(\boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_k) \ , & Tr(QP^s)\sum Q_j(\boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_k) & \end{array} \tag{11}$$

It is easy to check that none of the above terms changes the previous conclusions concerning the exactness of the CG equation. The only new quantity that enters produced by terms of type (11) is the expectation value of $\sum_i Q_i \sigma_{zi}$ on the baryons. For instance $Tr(QP^s)\sum Q_i(\boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_k) = -Q_B/2 + (3/2)\sum_i Q_i \sigma_{zi}$; the values of $\langle \sum_i Q_i \sigma_{zi} \rangle$ are listed in Table I of [7], indicated there as $\langle \Sigma_z^q \rangle$. One checks immediately that the CG equation remains true.

We conclude this discussion of the CG miracle as follows: The three quark terms are expected to give a very small, but non zero contribution to CG. It is remarkable that the hierarchy typical of the general parametrization, appears to

explain this smallness thus providing one of those few cases where one can estimate a tiny effect of the strong interactions and find it compatible with the data.

5. The Gell-Mann Okubo mass formula and its extension to the octet-decuplet

We will discuss now, first qualitatively, then quantitatively, the reason why, besides the CG formula, also the Gell-Mann Okubo (GMO) formula has its share of mysterious perfection. The GMO formula for the octet baryons:

$$\frac{1}{2}(n+\Xi^0)=\frac{1}{4}(3\Lambda+\Sigma) \tag{12}$$

is derived neglecting terms of second order in flavor breaking. The expansion parameter for flavor breaking is -as generally known and determined from the general parametrization [7, 8]- 0.3 to 0.33. Because the magnitude of first order flavor breaking is ≈ 150 to 190 MeV, an estimate of the contribution of second order flavor breaking terms -neglected in (12)- is ≈ 50 to 60 MeV. Instead the r.h.s and l.h.s of (12) differ by ≈ 6 MeV. Is this success just luck? Not entirely. In fact, by the general parametrization, we can now parametrize together the decuplet and octet masses and determine in this way all the 8 parameters in the octet+decuplet mass formula. The coefficient of the second order flavor breaking, $(a+b)$ in [5, 8] is in fact, as expected, ≈ 3 times smaller then the above estimate 50 or 60 MeV, due to the fact that necessarily second order flavor breaking terms have two indices. Thus one expects a difference of $\cong 17$ or 20 MeV between the left and right hand sides of (12). Because [5] the parameter that enters in the GMO formula is $(a+b)/2$ we are led, by this order of magnitude argument, to a difference of $8-10$ MeV, not far from the experimental value of ≈ 6 MeV. The essential role is once more played by the hierarchy, that again leads to negligible three index second order flavor breaking terms, precisely as for the CG formula.

Let us now be more quantitative. The fact that the coefficient c, d (Eq. (4,5) of [5]) multiplying the second order flavor three index terms are negligibly small was first discovered in relation to the GMO formula [5] (see also [8]) and used above in discussing the CG formula. Barring these c, d coefficients one finds a mass formula that relates the octet and decuplet masses, correct to second order in flavor, except for a three index term. This formula is just [5] the GMO formula plus a "decuplet" correction T:

$$\frac{1}{2}(n+\Xi^0)+T=\frac{1}{4}(3\Lambda+2\hat{\Sigma}^+-\Sigma^0) \tag{13}$$

where $\hat{\Sigma}^+\equiv 2\Sigma^+-\Sigma^0+2(p-n)$ and T is:

$$T=\Xi^{*-}-\frac{1}{2}(\Omega+\Sigma^{*-}) \tag{14}$$

The charge specification are inserted here because, at this accuracy, Eq.(13) must take into account the e.m. contributions: The combination (13) is unaffected by the e.m. corrections if the latter are calculated at zero order flavor breaking. (Note: Eq.(13) is more simply $\frac{1}{2}(p+\Xi^0)+T=\frac{1}{4}(3\Lambda+2\Sigma^+-\Sigma^0)$ but here we kept the form used in [5].)

While of course the improved Ξ^0 [1] slightly decreases the experimental error in (13), the main part of the error comes from the decuplet masses in T. With the conventional values of the masses [3] it is:

$$T(conventional)=5.18\pm 0.66 \text{ MeV}$$

whereas, if the pole values [3] of the resonances are taken [8], it is:

$$T(pole)=6.67\pm 1.25 \text{ MeV}$$

The left and right hand side of (13) become:

$$\begin{array}{lll} (conventional) & Left=1132.36\pm 0.7 \text{ MeV} & Right=1133.93\pm 0.04 \text{ MeV} \\ (pole) & Left=1133.86\pm 1.25 \text{ MeV} & Right=1133.93\pm 0.04 \text{ MeV} \end{array}$$

In both cases it is again true that three quark terms breaking flavor to second order are estimated to contribute to the difference between r.h.s. and l.h.s. less than 0.7 MeV.

We finally note that, to first order in $|m_u - m_d|/(\beta\Lambda_{QCD})$, the u, d mass difference does not affect the octet-decuplet mass formula (13); also the Trace terms do not modify the parametrization in this case.

6. Other e.m. mass formulas

Using the NRQM Gal and Scheck [6] derived long ago a set of relations among the electromagnetic masses for mesons and baryons. For mesons their assumptions were very restrictive, but for baryons they amounted mostly to the neglect of three body terms. In [2] we showed how these formulas could be reproduced by the general parametrization method under certain assumptions. Although it might be of interest to reanalyze the situation in more detail, we refrain from it because the experimental data did not change substantially.

However, in order to stimulate more precise measurements (if possible), we write down below a relation that is totally analogous, for the decuplet, to the Coleman Glashow formula for the octet and that can be treated in exactly the same way. It is independent of the Gal Scheck derivation, but easily verifiable using the Eqs.(27, 28) of [2]:

$$\delta\Delta^{+} - \delta\Delta^{0} = \delta\Sigma^{*+} - \delta\Sigma^{*-} + \delta\Xi^{*-} - \delta\Xi^{*0} \tag{15}$$

Incidentally Eq.(15), together with the relation (16) below, derived, as well known, from isospin algebra

$$\delta\Delta^{++} - \delta\Delta^{-} = 3(\delta\Delta^{+} - \delta\Delta^{0}) \tag{16}$$

might be of interest in connection with the determination of the mass differences

between the Δ's. More generally, Eq.(15) can be derived, using (16), from the charge corrected second order Okubo equations (7, 8) of ref.[14].

7. Conclusion

The reason for the extraordinary perfection of the Coleman-Glashow relation that holds at present to $\cong (0.29 \pm 0.25)/8 \cong (4 \pm 3) \cdot 10^{-2}$ (in spite of having been derived, we recall, using unbroken flavor SU(3)) is now clear. It depends, we have shown, on the smallness of the three index terms in the general parametrization. We underline again (Sect.4) that this smallness represents one of the few cases where, thanks to the hierarchy in the parametrization, an estimate of an effect due to the strong interaction can be given and found to be tiny as expected.

As to the Gell-Mann Okubo formula, its octet decuplet extension [5] including second order flavor breaking except for three index terms, holds to better than $2 \cdot 10^{-2}$. This value is the ratio between the experimental error (≈ 1 MeV) and what one would expect estimating the orders of magnitude (≈ 50 MeV) just by flavor breaking to second order. This confirms the smallness of the three index terms.

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